Revision 1(Solutions)

Year 11 Examination

Question/Answer Booklet

MATHEMATICS
SPECIALIST
UNITS 1 AND 2
Section One:
Calculator-free

Student Number: In figures | | | | | | | | |

In words

Teacher name

Time allowed for this section
Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section One: Calculator-free **35% (52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(7 marks)**

Two vectors are given by $\mathbf{a} = 9\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$. Determine

(a) a vector parallel to $\mathbf{a} - \mathbf{b}$ of magnitude 25. (3 marks)

$$\begin{bmatrix} 9 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\left\| \begin{bmatrix} 6 \\ 8 \end{bmatrix} \right\| = 10$$

$$\frac{25}{10} \times \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}$$

Find unit vector of $\underset{\sim}{a} - \underset{\sim}{b}$

$u = \underset{\sim}{a} - b$

$\hat{\underset{\sim}{u}} \times 25 =$

(b) $\mathbf{a}$ in terms of $\mathbf{d}$ and $\mathbf{e}$, where $\mathbf{d} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{e} = 5\mathbf{i} - 2\mathbf{j}$. (4 marks)

$$\begin{bmatrix} 9 \\ 4 \end{bmatrix} = x \begin{bmatrix} 3 \\ -5 \end{bmatrix} + y \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$3x + 5y = 9$

$-5x - 2y = 4$

$15x + 25y = 45$

$-15x - 6y = 12$

$19y = 57 \Rightarrow y = 3,\ x = -2$

$\mathbf{a} = -2\mathbf{d} + 3\mathbf{e}$

Question 2 **(7 marks)**

Three vectors are given by $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j}$.

Determine

(a) a unit vector $\mathbf{d}$, parallel to $\mathbf{a} + 2\mathbf{b}$. (3 marks)

Solution
$\mathbf{d} = \mathbf{a} + 2\mathbf{b} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$ and $\lvert\mathbf{d}\rvert = \sqrt{80} = 4\sqrt{5}$ $\hat{\mathbf{d}} = \frac{1}{4\sqrt{5}}\begin{bmatrix} 4 \\ -8 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$
Specific behaviours
✓ calculates $\mathbf{a} + 2\mathbf{b}$ ✓ calculates magnitude   ✓ states unit vector in simplified form

(b) the value(s) of k so that the magnitude of the vector $\mathbf{a} + k\mathbf{b}$ is 4. (4 marks)

Solution
$\mathbf{a} + k\mathbf{b} = \begin{bmatrix} 2 + k \\ -2 - 3k \end{bmatrix}$ Require $(2 + k)^2 + (-2 - 3k)^2 = 4^2$ $4 + 4k + k^2 + 4 + 12k + 9k^2 - 16 = 0$ $10k^2 + 16k - 8 = 0$ $5k^2 + 8k - 4 = 0$ $(5k - 2)(k + 2) = 0$ $k = \frac{2}{5}$ or $k = -2$
Specific behaviours
✓ writes magnitude equation   ✓ expands and simplifies equation   ✓ factorises equation   ✓ states both solutions

Question 3 **(9 marks)**

Consider the matrices $A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & -5 \end{bmatrix}$.

(a) It is possible to form the product of all four matrices. State the dimensions of the resulting product. (2 marks)

Solution
$ABDC$ or $BDAC$ are possible, both resulting in a 2×3 matrix.
Specific behaviours
✓ lists possible product ✓ states dimensions of product

(b) Determine the matrix $\frac{1}{2}DC$. (2 marks)

Solution
$\frac{1}{2} \times \begin{bmatrix} 4 & -5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 4 & -10 & 6 \end{bmatrix}$ $= \begin{bmatrix} 2 & -5 & 3 \end{bmatrix}$
Specific behaviours
✓ calculates DC ✓ calculates required result

(c) Determine the inverse of matrix A. (2 marks)

Solution
$A^{-1} = \frac{1}{8 - (-6)} \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$ $= \begin{bmatrix} 2 & 1.5 \\ 1 & 1 \end{bmatrix}$
Specific behaviours
✓ uses determinant ✓ determines inverse

(d) Clearly show use of matrix algebra to solve the system of equations $2x - 3y + 3 = 0$ and $4y = 2x + 2$. (3 marks)

Solution
$\begin{matrix} 2x - 3y = -3 \\ -2x + 4y = 2 \end{matrix} \Rightarrow AX = B$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $X = A^{-1}B = \begin{bmatrix} 2 & 1.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ $x = -3, y = -1$
Specific behaviours
✓ shows system can be written as matrix equation ✓ shows pre-multiplication of equation by inverse from (c) ✓ states solution of system

Question 4 **(7 marks)**

(a) Matrix A represents a rotation of 180º about the origin. Determine

(i) matrix A. (1 mark)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(ii) the exact coordinates of the point (-2, 3) after transformation by matrix A. (1 mark)

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \Rightarrow A'(2,-3)$$

(iii) the determinant of matrix A. (1 mark)

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(b) Matrix $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Describe the transformation represented by B and calculate its determinant. (2 marks)

B is a reflection in the y-axis.

$\det(B) = -1$

(c) Use an example to show that two non-singular square matrices C and D exist such that the determinant of their sum is equal to the sum of their determinants. (2 marks)

If $C = A$ and $D = B$ from above, then $\det(C) + \det(D) = -1 + 1 = 0$.

Also, $C + D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\det(C + D) = 0$.

Hence $\det(C + D) = \det(C) + \det(D)$.

Question 5 **(7 marks)**

(a) Solve the equation $\tan\left(\frac{x+25°}{2}\right) = \sqrt{3}$ for $0° \le x \le 540°$. (3 marks)

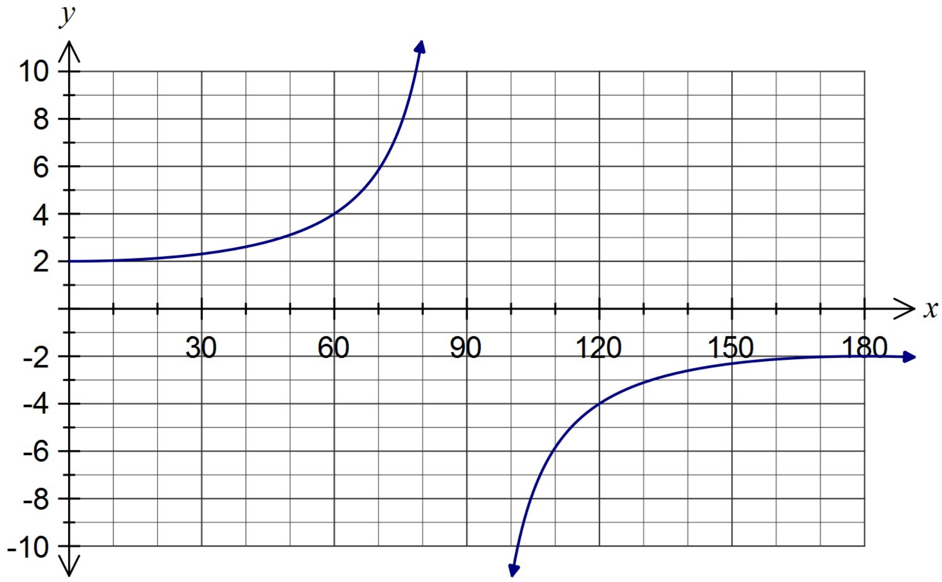
Solution
$0° \le x \le 540° \Rightarrow 12.5° \le \dfrac{x+25}{2} \le 282.5°$ $\dfrac{x+25°}{2} = 60°, 240°$ $x = 95°, x = 455°$
Specific behaviours
✓ uses $\tan 60° = \sqrt{3}$ ✓ determines first solution ✓ determines second solution

(b) Prove that $(1 - \cos x)(1 + \sec x) = \sin x \tan x$. (4 marks)

Solution
$LHS = 1 + \sec x - \cos x - \cos x \sec x$ $= \sec x - \cos x$ $= \dfrac{1 - \cos^2 x}{\cos x}$ $= \dfrac{\sin^2 x}{\cos x}$ $= \sin x \tan x$ $= RHS$
Specific behaviours
✓ expands and simplifies LHS ✓ combines into single fraction ✓ uses Pythagorean identity ✓ simplifies to RHS

Question 6 **(7 marks)**

(a) Sketch the graph of $y = 2\,\text{cosec}(x + 90)$ for $0° \le x \le 180°$. (3 marks)



(b) Prove the identity $\cot A + \tan A = \sec A\,\text{cosec}\,A$. (4 marks)

$$LHS = \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}$$

$$= \frac{\cos^2 A + \sin^2 A}{\sin A \cos A}$$

$$= \frac{1}{\cos A \sin A}$$

$$= \sec A\,\text{cosec}\,A$$

$$= RHS$$

Question 7 **(8 marks)**

(a) Prove that the sum of any three consecutive terms of an arithmetic sequence with first term a and common difference d is always a multiple of three, for $a, d \in \mathbb{N}$. (3 marks)

Solution
Let $T_n = a + (n-1)d$ so that $T_n + T_{n+1} + T_{(n+2)} = (a + nd - d) + (a + nd) + (a + nd + d)$ $= 3a + 3nd = 3(a + nd) \Rightarrow$ always a multiple of 3
Specific behaviours
✓ writes expression for three consecutive terms of arithmetic sequence ✓ simplifies expression ✓ factors 3 out and states conclusion

(b) Use mathematical induction to prove that $7^{2n-1} + 5$ is always divisible by 12, for $n \in \mathbb{N}$.

(5 marks)

Solution
Let $f(n) = 7^{2n-1} + 5$, so clearly true when $n = 1$ as $f(1) = 12$. Assume that $f(k)$ is always true, so that $f(k) = 7^{2k-1} + 5 = 12I$, where I is an integer. $f(k+1) = 7^{2(k+1)-1} + 5$ $= 7^{2+2k-1} + 5$ $= 7^2 \times 7^{2k-1} + 5$ $= 49 \times 7^{2k-1} + 5$ $= 48 \times 7^{2k-1} + 7^{2k-1} + 5$ (Using kth case) $= 48 \times 7^{2k-1} + 12I$ $= 12(4 \times 7^{2k-1} + I)$ Since $f(1)$ is divisible by 12, and it has been shown that if $f(k)$ is, so is $f(k+1)$, then $7^{2n-1} + 5$ is divisible by 12 for all $n \geq 1$.
Specific behaviours
✓ shows true for initial case ✓ assumes true for $n = k$ and equates result to multiple of 12 ✓ uses index laws to achieve $49 \times 7^{2k-1} + 5$ ✓ factors 12 out of expression ✓ makes summary statement